

differ from those presented in [1, 2]; this is associated with both the more precise definition of data on the limiting dynamic compression diagrams of the soils under consideration, and allowance for the peculiarities of their cyclic loading.

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METHOD OF ELASTIC CHARACTERISTIC VARIATION IN THE PROBLEM OF A LIMITING LOAD

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UDC 539.214:539.374

One of the questions of ideal plasticity theory is that of finding limiting loads with which a structure ceases to resist the action of external forces. Two-way evaluation of them may be obtained with the help of static and kinematic theorems [1]. Given below is a procedure based on these theorems making it possible to approach successively the upper and lower boundaries of the limiting load.

Let the condition for yielding have the form (the Mises-Hill criterion)

$$I = \sigma^T A \sigma = 1, \quad (1)$$

where σ is a vector-column composed of stress tensor components; A is a matrix of plastic flow characteristics; symbol T means the operation of transposing.

Equations for equilibrium within the body and at its boundary are written in operator form

$$D\sigma(x) = q(x), \quad q(x) = q_0(x)t. \quad (2)$$

Here D is a matrix of linear differential operators; $q_0(x)$ is normalized external load; t is loading parameter; x is radius-vector for a point of the body.

Coefficient t_* is sought on reaching which the structure loses its supporting capacity.

Lower Estimate. The solution of Eq. (2) is presented in the following symbolic form: $\sigma = \sigma_0 t$, $\sigma_0 = D^{-1}q_0$. We calculate function I : $I = I_0 t^2$, $I_0 = \sigma_0^T A \sigma_0$. Let with $t = t_-$ stress σ be reached for the flow surface of any point of the body. Then

$$(I_0)_{\max} t_-^2 = 1.$$

Since equilibrium equations are satisfied and stresses do not go beyond flow surface (1), then according to the static theorem

$$t_* \geq t_- = 1 / \sqrt{(I_0)_{\max}}. \quad (3)$$

As can be seen from (3), for the best estimate it is necessary to find the stress field minimizing $(I_0)_{\max_x}$ (see also [2, 3]). Thus it is necessary to find

$$I_* = \min_{\sigma_0} [(I_0)_{\max_x}]. \quad (4)$$

The main idea of the work [4] is that instead of field σ_0 two fields are found, i.e., strain field $\varepsilon = Lu$ (L is linear differential operator, u is displacement vector) and the matrix of elastic constants $E(x)$ of some theoretical Hooke's law $\sigma_0 = E\varepsilon$. The possibility of varying field E appears, e.g., to find it in the form

$$E = E_0(x)\lambda(x).$$

Here E_0 is an unchanged symmetrical matrix; $\lambda(x)$ is a variable scalar function of the radius-vector of a point of the body found from the condition of a minimum for the value $(I_0)_{\max_x}$.

A feature of $\lambda(x)$ is that it is sufficient to determine it with an accuracy up to a constant factor. In fact, the solution of Eq. (2) may be written as $\varepsilon = (DE_0\lambda)^{-1} q_0$. Then

$$I_0 = \lambda^2 [E_0(DE_0\lambda)^{-1} q_0]^T A [E_0(DE_0\lambda)^{-1} q_0].$$

The assertion stated above follows from this expression in view of the linearity of D .

We present the calculation operator $(I_0)_{\max_x}$ in the form [5]

$$(I_0)_{\max_x} = \lim_{p \rightarrow \infty} \left[(\text{mess } \Omega)^{-1/p} \left(\int_{\Omega} I_0^p d\Omega \right)^{1/p} \right] \quad (5)$$

($\text{mess } \Omega$) is a measure of body volume).

Reduction of the Control Problem to a Variational Problem. Let us consider the isoperimetric problem of variational reckoning of the minimization of functional $F(u, \lambda)$, where

$$F = \int_{\Omega} \varepsilon^T \lambda E_0 \varepsilon d\Omega - \int_{\Omega} Q_0^T u d\Omega - \int_{\Gamma} P_0^T u d\Gamma, \quad (6)$$

on condition that

$$(I_0)_{\max_x} = [\lambda^2 (E_0 \varepsilon)^T A (E_0 \varepsilon)]_{\max_x} = c^2 \quad (7)$$

(Q_0 and P_0 are normalized vectors of external forces; Γ is the surface of the body at which P_0 is prescribed; $u(x)$ is a kinematically possible displacement field; we vary field λ and u ; $c = \text{const}$). By substituting (5) in (7) and raising to power p we obtain instead of (7)

$$\lim_{p \rightarrow \infty} \int_{\Omega} I_0^p / c^{2p} d\Omega = \text{mess } \Omega. \quad (8)$$

Use of the Lagrangian multiple method leads to the problem of finding the stationary point for functional

$$\Phi = F + \mu \lim_{p \rightarrow \infty} \int_{\Omega} I_0^p / c^{2p} d\Omega. \quad (9)$$

Variation of (9) gives (furthermore $I_{00} = \varepsilon^T E_0 A E_0 \varepsilon$)

$$\int_{\Omega} \left[\varepsilon^T E_0 \varepsilon + \mu \lim_{p \rightarrow \infty} (2p \lambda^{2p-1} I_{00}^p / c^{2p}) \right] \delta \lambda d\Omega + \int_{\Omega} \delta \varepsilon^T \left[2 \lambda E_0 \varepsilon + \mu \lim_{p \rightarrow \infty} (2p \lambda^{2p-1} I_{00}^{p-1} E_0 A E_0 \varepsilon / c^{2p}) \right] d\Omega - \int_{\Omega} Q_0^T \delta u d\Omega - \int_{\Gamma} P_0^T \delta u d\Gamma = 0.$$

Whence

$$\lambda = \lim_{p \rightarrow \infty} \left[-\varepsilon^T E_0 \varepsilon I_{00}^{-p} c^{2p} / (2p\mu) \right]^{1/(2p-1)}. \quad (10)$$

It is possible to select p so that λ is greater than zero:

$$p = \begin{cases} (3 + 2n)/4, & \mu > 0, \\ (n + 1)/2, & \mu < 0 \quad (n = 1, 2, \dots). \end{cases}$$

Since energy $\varepsilon^T E_0 \varepsilon$ is limited almost everywhere, then from (10)

$$\lambda = I_{00}^{-1/2} \lim_{p \rightarrow \infty} (-c^{2p} \mu)^{1/(2p-1)}.$$

It is possible to select c^2 so that $(-c^{2p} \mu)^{-1/(2p-1)} = 1$. Then

$$\lambda = 1/\sqrt{I_{00}}. \quad (11)$$

In order to show that equilibrium equations emerge from the variational problem in question, it is sufficient to return to (6) by substituting (11):

$$F = \int_{\Omega} \varepsilon^T E_0 \varepsilon I_{00}^{-1/2} - \int_{\Omega} Q_0^T u \, d\Omega - \int_{\Gamma} P_0^T u \, d\Gamma.$$

Variation of F is written as

$$\delta F = \int_{\Omega} 2\varepsilon^T E_0 I_{00}^{-1/2} \delta \varepsilon \, d\Omega - \int_{\Omega} Q_0^T \delta u \, d\Omega - \int_{\Omega} \varepsilon^T E_0 \varepsilon \varepsilon^T E_0 A E_0 \delta \varepsilon I_{00}^{-3/2} \, d\Omega - \int_{\Gamma} P_0^T \delta u \, d\Gamma. \quad (12)$$

In view of symmetry of the expression $E_0 \varepsilon \varepsilon^T E_0 A E_0$, $\varepsilon^T E_0 \varepsilon \varepsilon^T E_0 A E_0 \delta \varepsilon = \varepsilon^T E_0 A E_0 \varepsilon \varepsilon^T E_0 \delta \varepsilon = I_{00} \varepsilon^T E_0 \delta \varepsilon$. From (12) taking account of the minimum of F

$$\delta F = \int_{\Omega} \varepsilon^T (E_0 \lambda) \delta \varepsilon \, d\Omega - \int_{\Omega} Q_0^T \delta u \, d\Omega - \int_{\Gamma} P_0^T \delta u \, d\Gamma = \int_{\Omega} \sigma_0^T \delta \varepsilon \, d\Omega - \int_{\Omega} Q_0^T \delta u \, d\Omega - \int_{\Gamma} P_0^T \delta u \, d\Gamma = 0. \quad (13)$$

Condition (13) conforms with the variational Lagrangian equation which is equivalent to an equilibrium equation. Furthermore, in view of the reciprocity principle problem (6), (7) is equivalent to the problem of minimizing $(I_0)_{\max}^x$ on the condition of constant

F with Euler equations which will also be equilibrium equations and condition (11). Consequently, problem (6), (7) is equivalent to control problem (2), (4), and the optimum condition will be equality (11).

Upper Estimate. By determining the displacement field from the equilibrium equation using elastic characteristics $E = \lambda E_0$ it is possible to find the upper boundary of the limiting load. According to the well-known theorem [1]

$$t_* \leq t_+ = W/U.$$

Here W is the cardinal number of plastic deformations; U is the cardinal number of external unit forces. As a kinematically possible velocity field $\dot{u}(x)$ it is possible to take field $u(x)$. Then it is easy to calculate W and U.

The procedure for approaching the boundaries of the limiting load is built up as follows. In order to find the initial approximation we take $\lambda^{(1)} = 1$. After solving Eq. (13) we find $\sigma_0^{(1)}$, $(I_0)_{\max}^{(1)}$, $W^{(1)}$, $U^{(1)}$ and make an estimate of the limiting load coefficient:

$$1/\sqrt{(I_0)_{\max}^{(1)}} \leq t_* \leq W^{(1)}/U^{(1)}. \quad (14)$$

If this estimate is not satisfactory, then in order to find the next estimate we take $\lambda^{(2)} = [(I_0)_{\max}^{(1)}]^{-1/2}$, solve Eq. (13) again, make estimate (14), etc.

Illustrative Example. The procedure for finding the upper boundary is demonstrated on the problem of estimating from below the supporting capacity of an element of a bent plate with thickness h. According to the Kirchhoff-Love hypothesis,

$$\sigma = E \varepsilon z, \quad \varepsilon^T = \{\kappa_{11}, \kappa_{22}, 2\kappa_{12}\}, \quad \sigma^T = \{\sigma^{11}, \sigma^{22}, \sigma^{12}\}, \quad (15)$$

where ε is a vector of curvature independent of coordinate z normal to the central surface of the plate; E is matrix of cylindrical stiffnesses. For the determination bending moments M^{11} , M^{22} , M^{12} are presented in the form

$$M = \int_{-h/2}^{h/2} \sigma z \, dz, \quad M^T = \{M^{11}, M^{22}, M^{12}\}. \quad (16)$$

Let $M = M_0 t$ be found from the equilibrium equation for the plate. Expression (16) may be interpreted as the integral equation of equilibrium. In the first step with $\lambda^{(1)} = 1$, $E = E_0$ from (15) and (16)

$$\varepsilon^{(1)} = E_0^{-1} M_0 l^2 / h^2.$$

From (11) and (15) it follows that $\lambda^{(2)} = \text{sign}(z)/z$. Substitution of $E = \lambda^{(2)} E_0$ in (15), and then in equilibrium, Eq. (16) gives

$$\sigma_0^{(2)} = 4M_0 \text{sign}(z)/h^2.$$

A normal two-layer model of the plate was obtained. Since $I_0 = \text{const}$ over the whole height, then

$$t_- = h^2 / (4\sqrt{M_0 A M_0}),$$

which is equivalent to the usual yielding condition used in ideal plastic plate theory.

Use of the Analytical Solutions. The procedure suggested makes it possible to use effectively analytical unilateral estimates (normally it is easy to obtain equations for the upper boundary). Then immediately it is possible to assume $\lambda^{(1)} = \sqrt{I_{00}}$, where I_{00} is calculated from the known solution.

As an example we consider the problem of the limiting load for a beam with length l with one fixed end and one hinged end (Fig. 1; m and R are reactions).

The method of plastic joints gives [1]:

$$q_+ = q_0 t_+ = 11,6 M_s, M_s = 2\sigma_s S_x / l^2 \quad (17)$$

(σ_s is yield point with axial tension, S_x is static moment).

Plastic hinges arise at the fixing ($z = 0$) and at distance $z = \xi l = 0.586l$ from it. By assuming that the beam consists of four sections with different stiffnesses ($E = E_1$ is assumed to be small in the first and third sections, and in the second and fourth $E = E_1/\alpha$, $\alpha \ll 1$), we find function u_i which approximates deflection u (see Fig. 1) in the section with number i :

$$\begin{aligned} E_1 J_x u_i(\xi) &= \Lambda_i [D_i + C_i \xi + f(\xi)], \Lambda_i = \begin{cases} 1, & i = 1, 3, \\ 0, & i = 2, 4, \end{cases} \\ \xi &= z/l, f(\xi) = l^2 m \xi^2 / 2 + l^3 \xi^3 R / 6 - l^4 q \xi^4 / 24, \\ D_1 &= C_1 = 0, C_2 = f'(\Delta) / \alpha, D_2 = (f(\Delta) - f'(\Delta)\Delta) / \alpha, \\ C_3 &= -f'(\xi), D_3 = f'(\xi)\xi - f(\xi), C_4 = [f'(\Delta) + f'''(\xi)\Delta] / \alpha, \\ D_4 &= -\xi f''(\xi)\Delta / \alpha, u_4(1) = u_4''(1) = 0 \end{aligned}$$

(J_x is moment of inertia, Δ is the relative length of sections 1 and 3). By letting α and Δ approach zero it is possible to estimate t_* . The maximum value of t_- is achieved with $\Delta/\alpha = 0.257$ and it conforms with (17).

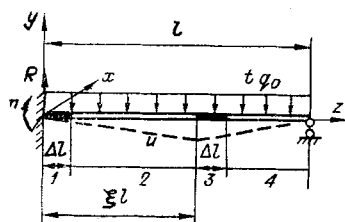


Fig. 1

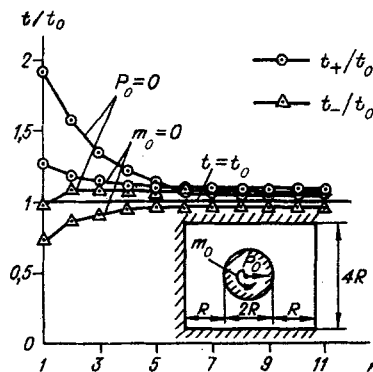


Fig. 2

Results of Numerical Experiments. Problem 1. The limiting value of the load applied through a rigid core in the plane of a plate was found (Fig. 2, t_0 is the upper boundary obtained in [6]).

Solution by the aforementioned procedure was performed by the finite element method (triangular elements with linear approximation of displacements was used). Calculations showed that after five to six iterations t_- and t_+ are almost stabilized. In addition, with $m = 0$ approximate value t_* calculated as the mean arithmetic value of t_+ and t_- stabilizes after the first approximation. For example, the difference between $t_* = (t_+ + t_-)/2$ in the second and ninth steps is 1.9%, and in the eighth and ninth it is -0.08% . The picture indicating convergence of the method in relation to the number of iterations is given in Fig. 2. Also given there are results for $P_0 = 0$ which also confirm rapid convergence of the method. A feature of this problem is the fact that the lower boundary is obtained quite well immediately.

Comments. Convergence undoubtedly depends on such parameters as Poisson's ratio, the maximum achieved value of $\eta = E_{\max}/E_{\min}$, and the degree of discretization.

Numerical experiments showed that first, the results depend little on Poisson's ratio and the difference is not more than 0.5%; second, the number n_* exists such that for a given degree of discretization with $\eta > \eta_*$ calculated results are almost independent of η , for example for 420 elements $n_* \approx 100$; third, with an insufficient degree of discretization the lower boundary may even appear above the upper boundary, which is known from another solution, e.g., analytical. This is a consequence of the fact that as a rule numerical methods smooth stress peaks, in view of which $(I_0)_{\max}$ obtains a lower true elasticity

modulus for the distribution adopted. This case occurs with $P = 0$ and is shown in Fig. 2.

Problem 2. The supporting capacity of a rectangular hinged plate under a uniform load was estimated. Triangular Zenkevich elements were used with cubic approximation of displacements in a 12×12 grid for a quarter of the region. Poisson's ratio was assumed to be 0.48.

Rapid stabilization of the mean arithmetic boundary for the limiting load was also observed here. Starting from the third iteration results differed from each other by not more than 2% and the mean value of limiting load after the fifth iteration with side ratios of 1 and 1.5 exceeded the analytical upper boundary by 4% and 3% respectively.

It is interesting to note that in the course of iteration the surface of a curved plate started to take the form of a pyramid which is used with kinematic analysis [7]. The kinematically permissible displacement field for a square plate [7] makes it possible to obtain a good estimate from below immediately.

The procedure suggested has a characteristic that makes it possible by means of well developed methods for solving problems of elasticity theory in a small number of iterations to approach the upper and lower boundaries of the limiting load. In this way individual results in solving the problem by analytical methods are used effectively.

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